Journal of Statistical Physics, Vol. 44, Nos. 1/2, 1986

On the Stationary Measures of Anharmonic Systems in the Presence of a Small Thermal Noise

J. Fritz[1]

Received October 3, 1985

We consider certain small stochastic perturbations of a d-dimensional infinite system of coupled anharmonic oscillators. The evolution law is reversible in the Yaglom sense, thus Gibbs states with the given interaction and temperature are stationary measures. If $d < 3$ then some stability properties of the interaction imply the converse statement; if $d > 2$ then the same is proven for translation invariant measures only. The methods and results of Ref. 4, 6–8 are extended to second-order systems of stochastic differential equations.

KEY WORDS: Continuous spin systems; interacting diffusions; Gibbs states; free energy; relative entropy; singular integrals.

1. INTRODUCTION

One of the most fundamental problems of statistical mechanics is certainly the description of the set of stationary states of the Hamiltonian dynamics of infinite classical systems. It is well known that every Gibbs state with the given interaction is in fact a stationary measure, but the converse statement is somewhat problematic. There exist some exactly solvable models as the free dynamics and the harmonic crystal, where the presence of some additional conservation laws gives rise to some new stationary states (see Ref. 13). The main purpose of this paper is to demonstrate that in the presence of certain arbitrarily small stochastic perturbations such a degeneration of the set of stationary states is not possible any more. Of course, we are not able to discuss this question in a full generality; we investigate lattice models with the simplest but most natural kind of stochastic perturbations.

[1] Mathematical Institute, HAS, H-1364 Budapest, POB 127, Hungary.

0022-4715/86/0700-0025$05.00/0

Let us consider a countable set S of one-dimensional oscillators of unit mass. Configurations of the system are represented as $\omega = (p_k, q_k)_{k \in S}$, where $p_k \in \mathbb{R}$ and $q_k \in \mathbb{R}$ denote the velocity and the position of oscillator $k \in S$. We are assuming that S is a connected graph, the set of neighbors of site $k \in S$ will be denoted by S_k; i.e., $j \in S_k$ and $k \in S_j$ are equivalent. Neighboring oscillators are interacting by a symmetric pair potential U, the system is stabilized by an external field, h, and each oscillator is connected to a thermal reservoir of temperature $T > 0$ and damping $\lambda > 0$; i.e., the evolution law is given by the system of stochastic differential equations

$$\begin{aligned} dp_k &= -h'(q_k)\,dt - \sum_{j \in S_k} U'(q_k - q_j)\,dt - \lambda p_k\,dt + \sqrt{2\lambda T}\,dw_k \\ dq_k &= p_k\,dt, \qquad k \in S \end{aligned} \tag{1}$$

where w_k, $k \in S$ is a family of independent, standard Wiener processes, h' and U' denote the derivatives of h and U. Infinite systems of this kind are discussed in Ref. 2, 3, 15. The problem of existence and uniqueness of solutions to (1) can be solved essentially in the same way as for Hamiltonian systems.[2] We are not going to prove the best existence theorem, our conditions are subordinated to the problem of stationary measures. Under some natural conditions on h and U, (1) defines a Markov process, $\mathbb{P}^t$ in a space $\Omega \subset (\mathbb{R}^2)^S$ of allowed configurations to be specified later. This configuration space is so large that $\mu(\Omega) = 1$ for a wide class of probability measures including the Gibbs states for the given interaction. Introduce

$$H_k(\omega) = \tfrac{1}{2}p_k^2 + h(q_k) + \sum_{j \in S_k} U(q_k - q_j) \tag{2}$$

the energy of the oscillator at site $k \in S$, then μ is a Gibbs state with potential h and U at temperature T if the joint conditional density of p_k and q_k given p_j and q_j for $j \neq k$ is proportional to $\exp(-1/T\, H_k(\omega))$ for each $k \in S$. The formal generator $\mathbb{G}$ of $\mathbb{P}^t$ can be written as $\mathbb{G} = \mathbb{L} + \lambda \mathbb{G}_p$, where

$$\mathbb{L}\varphi = \sum_{k \in S} [p_k \nabla_{qk} \varphi - (\nabla_{qk} H_k) \nabla_{pk} \varphi] \tag{3}$$

is the Liouville operator, which is conservative, and

$$\mathbb{G}_p \varphi = \sum_{k \in S} (T\nabla_{pk}^2 \varphi - p_k \nabla_{pk} \varphi) \tag{4}$$

is a dissipative generator, while ∇_{pk} and ∇_{qk} denote differentiation with respect to p_k and q_k. Let $\mathbb{C}_0^2$ denote the space of local functions

$\varphi: (\mathbb{R}^2)^S \to \mathbb{R}$ with two continuous derivatives, then $\varphi \in \mathbb{C}_0^2$ implies $\nabla_{pk}\varphi = \nabla_{qk}\varphi = 0$ for all but a finite number of $k \in S$, thus $\mathbb{G}$ is well defined on $\mathbb{C}_0^2$.

Suppose now that μ is a Gibbs state for h and U with temperature $T > 0$, and observe that for $\varphi \in \mathbb{C}_0^2$ we have

$$\mathbb{L}\varphi = T \sum_{k \in S} e^{H_k/T}[\nabla_{qk}(e^{-H_k/T}\nabla_{pk}\varphi) - \nabla_{pk}(e^{-H_k/T}\nabla_{qk}\varphi)] \tag{5}$$

$$\mathbb{G}_p\varphi = T \sum_{k \in S} e^{H_k/T}\nabla_{pk}(e^{-H_k/T}\nabla_{pk}\varphi) \tag{6}$$

thus integrating by parts we obtain for $\varphi_1, \varphi_2 \in \mathbb{C}_0^2$ that

$$\begin{aligned}\int \varphi_1 \mathbb{G}\varphi_2 \, d\mu = {} & -\lambda T \sum_{k \in S} \int (\nabla_{pk}\varphi_1)(\nabla_{pk}\varphi_2)\, d\mu \\ & + T \sum_{k \in S} \int [(\nabla_{pk}\varphi_1)(\nabla_{qk}\varphi_2) - (\nabla_{qk}\varphi_1)(\nabla_{pk}\varphi_2)]\, d\mu \end{aligned} \tag{7}$$

provided that the expectations make sense. In contrast to the usual reversibility property

$$\int \varphi_1 \mathbb{G}\varphi_2 \, d\mu = \int \varphi_2 \mathbb{G}\varphi_1 \, d\mu \tag{8}$$

introduced by Kolmogorov[9] and often referred to as the principle of microscopic balance, (7) is obviously equivalent to the reversibility property

$$\int \varphi_1 \mathbb{G}\varphi_2 \, d\mu = \int \varphi_2^* \mathbb{G}\varphi_1^* \, d\mu \tag{9}$$

introduced by Yaglom [16], where $\varphi^*(\omega) = \varphi(\omega^*)$ and $\omega^* = (-p_k, q_k)_{k \in S}$ if $\omega = (p_k, q_k)_{k \in S}$. Notice that Hamiltonian systems satisfy (9). Putting $\varphi_2 = 1$ we see that both (8) and (9) imply the stationary Kolmogorov equation

$$\int \mathbb{G}\varphi \, d\mu = 0 \qquad \text{for} \quad \varphi \in \mathbb{C}_0^2 \tag{10}$$

whenever the expectation makes sense. More intuitively, (9) means that the probability measure of the corresponding equilibrium process is symmetric with respect to the reflection $\omega(t) \to \omega^+(-t)$ of trajectories; (8) is related to the symmetry $\omega(t) \to \omega(-t)$. Let us remark that (1) seems to be the simplest stochastic system of interacting components which is reversible in the Yaglom sense. The formal generator of a wider class of processes satisfying (9) decomposes as $\mathbb{G} = \mathbb{L} + \mathbb{G}_p + \mathbb{G}_q$, where

$$\mathbb{L}\varphi = T \sum_{k \in S} e^{H_k/T}[\nabla_{qk}(F_k e^{-H_k/T} \nabla_{pk}\varphi) - \nabla_{pk}(F_k e^{-H_k/T} \nabla_{qk}\varphi)] \tag{11}$$

$$\mathbb{G}_p\varphi = \frac{T}{2} \sum_{k \in S} e^{H_k/T} \nabla_{pk}(a_k^2 e^{-H_k/T} \nabla_{pk}\varphi) \tag{12}$$

$$\mathbb{G}_q\varphi = \frac{T}{2} \sum_{k \in S} e^{H_k/T} \nabla_{qk}(b_k^2 e^{-H_k/T} \nabla_{qk}\varphi) \tag{13}$$

F_k, a_k, and b_k are arbitrary symmetric functions, i.e., $F_k^* = F_k$, $a_k^* = a_k$, and $b_k^* = b_k$.

The idea that one can use information-theoretical methods to prove ergodic theorems for Markov processes goes back to Rényi.[12] The main point is that the measures of information like the relative entropy with respect to the equilibrium state are monotonic functions of time. In the case of infinite systems only the local versions of such quantities make sense, and they are not monotonic anymore. Nevertheless, under the additional condition of reversibility in the Kolmogorov sense, Holley[6] managed to control the relative entropy (i.e. the free energy) of the finite-dimensional distributions of stochastic Ising models in translation invariant situations; he proved that every translation invariant stationary measure is a Gibbs state for the given interaction and temperature. This method and result was extended to nontranslation invariant stationary measures in dimensions one and two by Holley and Stroock.[7] For the treatment of some continuous spin systems (stochastic Heisenberg models) see Holley and Stroock[8] and Fritz.[4] Here we are going to adapt this technique of free energy to interacting diffusions which are reversible only in the Yaglom sense. The basic ideas are quite easy. Since $e^{t\mathbb{L}}$ is a group, the Hamiltonian part of the dynamics preserves the relative entropy with respect to any equilibrium state, and the free energy is just a linear function of the relative entropy with respect to the equilibrium Gibbs state. On the other hand, the dissipative component, $e^{t\mathbb{G}_p}$ is reversible in the Kolmogorov sense, but it is acting in the space of velocities only. Therefore we expect that the free energy decreases unless the state we consider has a Maxwellian distribution of velocities, whence the Gibbs property follows by the stationary Kolmogorov equation (10). This argument is trivial for finite systems, in the case of infinite systems we have to understand that every stationary measure admits smooth local densities, and some boundary effects should be controlled, too. Since $\mathbb{G}$ is a degenerated elliptic operator, the Malliavin calculus of Ref. 8 does not work in our case; the regularization trick of Ref. 4 will be used. The crucial step of the proof is the choice of the singular integral defining the approximating family of local densities. As in Reference 4, all calculations are based on the existence of some moments. Under some more or less natural stability properties of the interaction we prove that every stationary measure satisfies the moment conditions we need.

1. MAIN RESULT

First we summarize our conditions on S, h, and U. For each $k \in S$ we define a sequence of boxes centered at k by $\Lambda_1(k) = \{k\} \cup S_k$, $\Lambda_n(k) = \cup \Lambda_1(j)$ for $j \in \Lambda_{n-1}(k)$ if $n > 1$, the boundary of $\Lambda_n(k)$ is then defined as $B_n(k) = \Lambda_n(k) \backslash \Lambda_{n-1}(k)$; $B_1(k) = S_k$. If θ is a distinguished element of S, then $|k|$ denotes the distance of $k \in S$ from θ, i.e., $|k| = \min n$ such that $k \in S_n(\theta)$. We are assuming that

$$\sup_{k \in S} \operatorname{card} S_k < +\infty \tag{1.1}$$

$$\lim_{n \to \infty} \sup_{m \in S} \frac{\operatorname{card} B_n(m)}{\operatorname{card} \Lambda_n(m)} = 0 \tag{1.2}$$

$$\sum_{n=1}^{\infty} [\operatorname{card} B_n(\theta)]^{-1} = +\infty \tag{1.3}$$

Notice that (1.2) implies $\lim e^{-\varepsilon n} \operatorname{card} \Lambda_n(\theta) = 0$ for each $\varepsilon > 0$ as $n \to \infty$. If $S = \mathbb{Z}^d$, the d-dimensional integer lattice, and S_k contains the nearest neighbors of k, then (1.3) is possible only if $d \leqslant 2$. In translation invariant situations (1.3) is not needed.

Our conditions on the interaction are the following. The potential functions $h: \mathbb{R} \to [0, \infty)$ and $U: \mathbb{R} \to [0, \infty)$ are assumed to have two continuous derivatives, $U(x) = U(-x)$, and we have some $a > 0$ and $b \geqslant 0$ such that $xh'(x) \geqslant -b$, $xU'(x) \geqslant -b$, and

$$x^2 + y^2 + U'^2(x-y) \leqslant a[1 + 2b + xh'(x) + yh'(y)] \tag{1.4}$$

$$\begin{aligned}&\left[\frac{h'(x) - h'(\bar{x})}{x - \bar{x}}\right]^2 + \left[\frac{U'(x-y) - U'(\bar{x} - \bar{y})}{|x - \bar{x}| + |y - \bar{y}|}\right]^2 \\ &\qquad \leqslant a[1 + h(x) + h(\bar{x}) + h(y) + h(\bar{y})],\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\left[\frac{h'(x) - h'(\bar{x})}{|x - \bar{x}| + |x - \bar{x}|^2}\right]^2 + \left[\frac{U'(x-y) - U'(\bar{x} - \bar{y})}{|x - \bar{x}| + |y - \bar{y}| + |x - \bar{x}|^2 + |y - \bar{y}|^2}\right]^2 \\ &\qquad \leqslant a[1 + 4b + xh'(x) + yh'(y) + \bar{x}h'(\bar{x}) + \bar{y}h'(\bar{y})]\end{aligned} \tag{1.6}$$

These conditions are automatically fulfilled if h'' and U'' are bounded, and $\liminf h''(x) > 0$ as $x \to \pm\infty$. Notice that both $h''(x)$ and $U'(x)$ are necessarily bounded by a quadratic function of x.

We shall show that (1) defines a Markov process $\mathbb{P}^t$ in the configuration space Ω defined as

$$\Omega = \left[\omega \in (\mathbb{R}^2)^S\colon \sup_{k \in S} \frac{1 + h(q_k) + p_k^2}{1 + |k|^\sigma} < +\infty \quad \text{for some} \quad \sigma < 4 \right] \tag{1.7}$$

Equip Ω with the product topology and the associated Borel field, and let $\mathbb{C}_{0b}^2 \subset \mathbb{C}_0^2$ denote the space of bounded cylinder functions with continuous and bounded first and second derivatives. If $\varphi\colon \Omega \to \mathbb{R}$ is measurable and bounded, then $\mathbb{P}^t\varphi = \mathbb{P}^t\varphi(z)$ denotes the conditional expectation of $\varphi(\omega(t))$ given $\omega(0) = z$, where $\omega(t)$ is the process defined by (1) in a sense to be specified in the next section. The minimal requirement concerning $\mathbb{P}^t$ is that $\mathbb{B}$, the space of bounded measurable functions $\varphi\colon \Omega \to \mathbb{R}$ is mapped into itself by $\mathbb{P}^t$. There are several methods to construct solutions to (1). Here we follow Ref. 2, 3 and prove

Theorem 1. Suppose (1.1), 1.2) and (1.4), (1.5), then there exists a transition semigroup $\mathbb{P}^t\colon \mathbb{B} \to \mathbb{B}$ such that for $\varphi \in \mathbb{C}_{0b}^2$ we have

$$\mathbb{P}^t\varphi = \varphi + \int_0^t \mathbb{P}^s \mathbb{G}\varphi \, ds \quad \blacksquare$$

The conditions of this theorem are far from being optimal; as a matter of fact, a hierarchy of existence theorems can be imagined. If (1.4) and (1.5) are weakened, then Ω, the space of allowed configurations, gets smaller (see Ref. 2). On the other hand, assuming that h'' and U'' are bounded, we can allow initial configurations with an exponential growth rate (cf. Ref. 3). The reason for presenting Theorem 1 is economy of the paper. To prove that every stationary measure satisfies certain moment conditions, we need an a priori bound, and this a priori bound yields Theorem 1 at the same time. Although we have a uniqueness result for strong solutions to (1), we do not know that the Markov process of Theorem 1 is uniquely determined by the Kolmogorov equation given there. Of course, this equation makes sense only if $\mathbb{P}^t$ satisfies some moment conditions. Therefore on the Markov process defined by (1) we always mean the process to be constructed in Section 2, where the moment conditions mentioned above are verified, too. Let us emphasize, however, that this preferred process is not arbitrary; it will be constructed as the limit of the associated finite-dimensional processes.

Theorem 2. Let $\lambda > 0$, $T > 0$ and suppose (1.1)–(1.6), then every stationary measure of $\mathbb{P}^t$ is a Gibbs state for U and h at temperature T. ∎

This result can be formulated without any reference to the dynamics. One can replace the condition of stationarity by its consequence (10) for $\varphi \in \mathbb{C}^2_{0b}$, provided that all coefficients of $\mathbb{G}$ have finite expectations. The structure of the proof can be outlined as follows. Introduce

$$z_n(\omega) = \exp\left(-\frac{1}{T} \sum_{k \in \Lambda_n} \left[\frac{1}{2} p_k^2 + h(q_k) + \frac{1}{2} \sum_{j \in S_k \cap \Lambda_n} U(q_k - q_j) \right] \right) \tag{1.8}$$

where $\Lambda_n = \Lambda_n(\theta)$, and for any probability measure μ on Ω let

$$u_n(\omega) = \int g_n(\omega, \bar{\omega})\, \mu(d\bar{\omega}) \tag{1.9}$$

where

$$g(p, q) = c\sigma(1 + \sigma p^2 + \sigma q^2)^{-9}, \quad \iint g(p, q)\, dp\, dq = 1$$

and

$$g_n(\omega, \bar{\omega}) = \prod_{k \in \Lambda_n} g(p_k - \bar{p}_k, q_k - \bar{q}_k)$$

$\bar{p}_k$ and $\bar{q}_k$ denote the coordinates of $\bar{\omega} \in \Omega$, i.e., $\bar{\omega} = (\bar{p}_k, \bar{q}_k)_{k \in S}$. Since g_n is a singular integral as $\sigma \to +\infty$

$$I_n = \int u_n(\omega) \log \frac{u_n(\omega)}{z_n(\omega)}\, d_n\omega = \iint g_n(\omega, \bar{\omega}) \log \frac{u_n(\omega)}{z_n(\omega)}\, d_n\omega\, \mu(d\bar{\omega}) \tag{1.10}$$

approximates the free energy of μ in Λ_n; here and later the abbreviation

$$d_n\omega = \prod_{k \in \Lambda_n} dp_k\, dq_k$$

is used. Suppose now that $\mu = \mu_t$ evolves with time according to (1), then $du_n/dt = \int \bar{\mathbb{G}} g_n(\omega, \bar{\omega})\, \mu_t(d\bar{\omega})$, where the bar indicates that the generator is acting on g_n as a function of $\bar{\omega}$, thus

$$\frac{dI_n}{dt} = \iint \log \frac{u_n(\omega)}{z_n(\omega)}\, \bar{G} g_n(\omega, \bar{\omega})\, \mu_t(d\bar{\omega})\, d_n\omega \tag{1.11}$$

provided that the expectation makes sense. Integrating by parts we obtain

$$\frac{dI_n}{dt} = -\lambda T \sum_{k \in \Lambda_n} F_k(n) + T \sum_{k \in B_n} G_k(n) + R(n, \sigma) \tag{1.12}$$

where $B_n = B_n(\theta)$ is the boundary of Λ_n, $\lim R(n, \sigma) = 0$ as $\sigma \to +\infty$, and the dominant terms read as

$$F_k(n) = \int u_n(\omega) \left[\nabla_{pk} \log \frac{u_n(\omega)}{z_n(\omega)} \right]^2 d_n\omega \tag{1.13}$$

Therefore, if μ is a stationary measure, then we expect that $F_k(n) \to 0$ as $\sigma \to +\infty$, i.e., μ has Maxwellian velocities of temperature T, whence the Gibbs property of μ follows by an additional, but easy calculation via (10). In order to let this argument work on a rigorous level, we assume first that

$$\sup_{k \in S} \int [p_k^2 + q_k h'(q_k)] \, \mu(d\omega) < +\infty \tag{1.14}$$

then (10) makes sense for all $\varphi \in \mathbb{C}_{0b}^2$, thus we have

$$\iint f\left(\frac{u_n(\omega)}{z_n(\omega)}\right) \bar{G}g_n(\omega, \bar{\omega}) \, \mu(d\bar{\omega}) \, d_n\omega = 0 \tag{1.15}$$

whenever f: $\mathbb{R} \to \mathbb{R}$ is bounded. Now, following Ref. 4, we can integrate by parts, and letting f go to $\log x$ we conclude that $\lim F_k(n) = 0$ as $\sigma \to +\infty$, which implies the statement. Finally, exploiting (1.4) we show that every stationary measure satisfies both (1.14) and (1.15), which will complete the proof.

Just as in the case of stochastic Ising models (see Ref. 7) (1.3) can be relaxed in translation invariant situations only.

Theorem 3. Let $\lambda > 0$, $T > 0$, suppose (1.1), (1.2), (1.4), (1.5), (1.6), and let S be a group with neutral element θ such that $S_k = kS_\theta$ for each $k \in S$. If μ is a stationary measure of $\mathbb{P}^t$, and μ is invariant under all translations T_m, $m \in S$ defined as $T_m\omega = (p_{mk}, q_{mk})_{k \in S}$ if $\omega = (p_k, q_k)_{k \in S}$, then μ is a Gibbs state for U and h with temperature T. ∎

Since (1.1), (1.2), (1.4), and (1.5) are conditions of all three theorems, their validity will always be assumed in the forthcoming proofs.

2. CONSTRUCTION OF THE STOCHASTIC DYNAMICS

In this section we derive some a priori bounds implying the existence of the Markov process we are going to investigate later. The very same technique yields (1.14) for arbitrary stationary measures of $\mathbb{P}^t$. These calculations are based on the following couple of Liapunov functions (cf. Ref. 3, 4). For $m \in S$, $\omega \in \Omega$ and $\delta > 0$ let

$$Q_m(\omega)=\sum_{k\in S} e^{-\delta\rho(k,m)}\left[\frac{1}{2}p_k^2+h(q_k)+\frac{1}{2}\sum_{j\in S_k} U(q_k-q_j)\right]$$
$$+\frac{\lambda}{2}\sum_{k\in S} e^{-\delta\rho(k,m)}\left[p_k q_k+\frac{\lambda}{2}q_k^2\right] \tag{2.1}$$

$$W_m(\omega)=\sum_{k\in S} e^{-\delta\rho(k,m)}[1+b+p_k^2+q_k h'(q_k)] \tag{2.2}$$

where $\rho(k,m)$ denotes the distance of k and m in S, i.e., $\rho(k,m)=\min n$ with $k\in\Lambda_n(m)$. In view of (1.4), $U(x-y)$ is bounded by a linear function of $h(x)$ and $h(y)$, thus both Q_m and W_m are well defined on Ω for each $\delta>0$. Moreover, $\omega\in\Omega$ if, and only if

$$\sup_{m\in S}\frac{Q_m(\omega)}{1+|m|^\sigma}<+\infty \qquad \text{for some } \sigma<4 \tag{2.3}$$

Indeed, if $\omega\in\Omega$ then we have some $C>0$ and $\sigma<4$ such that

$$Q_m(\omega)\leqslant C\sum_{k\in S} e^{-\delta\rho(k,m)}(1+|k|^\sigma)$$
$$\leqslant 2^\sigma C(1+|m|^\sigma)\sum_{k\in S}(1+\rho^\sigma(k,m))\,e^{-\delta\rho(k,m)}$$

whence (2.3) follows by (1.2); the converse statement is trivial.

Lemma 1. We have a universal constant K such that

$$\mathbb{G}Q_m(\omega)\leqslant\left[\delta K(1+\lambda)\,e^\delta-\frac{\lambda}{2}\right]W_m(\omega)+\lambda(T+K)\sum_{k\in S} e^{-\delta\rho(k,m)}$$

for all $\omega\in\Omega$, $m\in S$, and $\delta>0$. ∎

Proof. If $\omega\in\Omega$ then the terms of $\mathbb{G}Q_m$ can be rearranged, and a direct calculation yields

$$\mathbb{G}Q_m=\frac{\lambda}{2}\sum_{k\in S} e^{-\delta\rho(k,m)}(2T-p_k^2-q_k h'(q_k))$$
$$+\frac{1}{4}\sum_{k\in S}\sum_{j\in S_k}[e^{-\delta\rho(j,m)}-e^{-\delta\rho(k,m)}](p_k+p_j)\,U'(q_k-q_j)$$
$$+\frac{1}{4}\sum_{k\in S}\sum_{j\in S_k}[q_j e^{-\delta\rho(j,m)}-q_k e^{-\delta\rho(k,m)}]\,U'(q_k-q_j)$$

Observe now that

$$|e^{-\delta\rho(j,m)} - e^{-\delta(k,m)}| \leqslant \delta \qquad \text{and} \qquad e^{-\delta\rho(j,m)} \leqslant e^{\delta} e^{-\delta\rho(k,m)}$$

if $j \in S_k$, while

$$2q_j e^{-\delta\rho(j,m)} - 2q_k e^{-\delta\rho(k,m)} = (q_j + q_k)[e^{-\delta\rho(j,m)} - e^{-\delta\rho(k,m)}] + (q_j - q_k)[e^{-\delta\rho(j,m)} + e^{-\delta\rho(k,m)}]$$

thus comparing

$$2\left| U'(q_k - q_j)\left(p_k + p_j + \frac{\lambda}{2} q_k + \frac{\lambda}{2} q_j\right)\right| \leqslant (1+\lambda)\, U'^2(q_k - q_j) + p_k^2 + p_j^2 + \frac{\lambda}{2} q_k^2 + \frac{\lambda}{2} q_j^2$$

and $-(q_k - q_j)\, U'(q_k - q_j) \leqslant b$, we obtain the statement as a direct consequence of (1.4).

Our a priori bound for the construction of solutions is the following consequence of Lemma 1.

Lemma 2. Suppose that S is a finite set, and $\delta > 0$ is small enough, then for each strong solution $\omega = \omega(t)$ to (1) we have a family R_m, $m \in S$ of random variables such that $P[R_m > c + u] \leqslant e^{-u}$ and

$$\sup_{s \leqslant t} Q_m(\omega(s)) \leqslant [1 + Q_m(\omega(0)) + R_m]\, e^{ct}$$

for all $t > 0$ and $m > 0$, where the constant c does not depend on S, ω, and m. ∎

Proof. The stochastic differential of $Q_m(\omega(t))$ reads as

$$dQ_m = \mathbb{G} Q_m\, dt + \sum_{k \in S} r_k\, dw_k$$

where

$$r_k = \sqrt{2\lambda T}\, \nabla_{pk} Q_m = \sqrt{2\lambda T}\, e^{-\delta\rho(k,m)}\left(p_k + \frac{\lambda}{2} q_k\right) \tag{2.4}$$

Therefore, if $2\lambda K(1+\lambda)\, e^{\delta} \leqslant \lambda$, then Lemma 1 implies

$$\begin{aligned} de^{-ct}(1 + Q_m) &= -ce^{-ct}(1 + Q_m)\, dt + e^{-ct}\, dQ_m \\ &\leqslant \lambda(T + K)\, e^{-ct} \sum_{k \in S} e^{-\delta\rho(k,m)}\, dt + \tfrac{1}{2} e^{-2ct} r_k^2\, dt \\ &\quad + \sum_{k \in S} (e^{-ct} r_k\, dw_k - \tfrac{1}{2} e^{-2ct} r_k^2\, dt) \end{aligned}$$

Introduce now

$$R_m = c + \sup_{t>0} \sum_{k \in S} \int_0^t (e^{-ct} r_k \, dw_k - \tfrac{1}{2} e^{-2ct} r_k^2 \, ds)$$

and notice that (1.2) implies $\sum e^{-\delta\rho(k,m)} \leqslant c_1$, while (1.4) yields $\sum r_k^2 \leqslant c_2 + c_3 Q_m$, which implies the inequality of Lemma 2; the bound on the tail of R_m is just the favorite maximal inequality of Ref. 11.

Now we are in a position to construct solutions to the infinite system (1).

Proposition 1. In the special case of $S = \Lambda_n(\theta)$ we denote by $\omega^{(n)} = \omega^{(n)}(t, z)$ the strong solution to (1) with initial condition $\omega^{(n)}(0, z) = z \in Q$. Then the limiting process $\omega = \omega(t, z)$ exists in the sense that

$$P[\lim_{n \to \infty} \sup_{s \leqslant t} (|p_k^{(n)}(s, z) - p_k(s, z)| + |q_k^{(n)}(s, z) - q_k(s, z)|) = 0] = 1$$

for all $t > 0$, $k \in S$ and $z \in \Omega$, where $\omega^{(n)} = (p_k^{(n)}, q_k^{(n)})_{k \in \Lambda_n(\theta)}$. This limiting solution is actually a strong solution to (1) satisfying the initial condition $\omega(0, z) = z$ as well as the conclusion of Lemma 2, and there is no other solution with these two properties. ▮

Proof. Introduce

$$d_k^{(n)}(t) = \max_{s \leqslant t} e^{-\lambda s} |p_k^{(n+1)}(s, z) - p_k^{(n)}(s, z)|$$

and

$$D_m^{(n)}(t, r) = \max_{k \in \Lambda_r(m)} d_k^{(n)}(t)$$

Let $k \in \Lambda_{n-1}(\theta)$, then (1) yields

$$e^{-\lambda t} |p_k^{(n+1)}(t, z) - p_k^{(n)}(t, z)| \leqslant \int_0^t |h'(q_k^{(n+1)}) - h'(q_k^{(n)})| \, e^{-\lambda s} \, ds$$

$$+ \sum_{j \in S_k} \int_0^t |U'(q_k^{(n+1)} - q_j^{(n+1)}) - U'(q_k^{(n)} - q_j^{(n)})| \, e^{-\lambda s} \, ds$$

and

$$e^{-\lambda t} |q_k^{(n+1)}(t, z) - q_k^{(n)}(t, z)| \leqslant e^{-\lambda t} \int_0^t |p_k^{(n+1)} - p_k^{(n)}| \, ds$$

$$\leqslant e^{-\lambda t} \int_0^t e^{\lambda s} d_k^{(n)}(s) \, ds \leqslant \int_0^t d_k^{(n)}(s) \, ds$$

as $\lambda > 0$, thus comparing (1.5) and Lemma 2 we obtain that

$$d_k^{(n)}(t) \leqslant L e^{ct/2}[1 + Q_k(z) + R_k^{(n)} + R_k^{(n+1)}]^{1/2} \int_0^t (t-s)\, D_k^{(n)}(s, 1)\, ds$$

where L is a universal constant, $R_m^{(n)}$ denotes the random variable associated with $\omega^{(n)}$ in Lemma 2. Therefore as $z \in \Omega$, we have a $\sigma < 4$ such that for $r < n-2$

$$D_\theta^{(n)}(t, r) \leqslant L_n e^{ct/2}(1+r)^{\sigma/2} \int_0^t (t-s)\, D_\theta^{(n)}(s, r+1)\, ds$$

where

$$L_n = \max_{m \in \Lambda_n(\theta)} L[1 + Q_m(z) + R^{(n)} + R^{(n+1)}]^{1/2}(1 + |m|)^{-\sigma/2}$$

and with some $\bar{c} = \bar{c}(z)$ we have

$$P[L_n > \bar{c} + u] \leqslant 2 \sum_{m \in \Lambda_n(\theta)} \exp(-u(1 + |m|^{\sigma/2})$$

This inequality can be iterated $n - r - 2$ times, and we get

$$D_\theta^{(n)}(t, r) \leqslant [L_n e^{ct/2}]^n \frac{(n!)^{\sigma/2}}{(2n-2r-4)!}$$

Since $\sigma < 4$ and $\sigma \geqslant 1$ may be assumed, an easy calculation results in

$$P\left[\sum_{n=1}^{\infty} d_k^{(n)}(t) < +\infty\right] = 1$$

for all $t > 0$ and $k \in S$, which proves that $\omega^{(n)} \to \omega$ as $n \to +\infty$. The integral form of (1) and the estimate of Lemma 2 follow by a straightforward procedure via Lemma 2, while the uniqueness result can be proven by means of the above iteration method with the simplification that the number of allowed iterations is not limited.

Proof of Theorem 1. As a limit of solutions to finite systems, the general solution, $\omega = \omega(t, z)$ is a jointly measurable function of $t > 0$ and $z \in \Omega$, thus $\mathbb{P}_\varphi^t$ is well defined and it is measurable for each measurable and bounded $\varphi : \Omega \to \mathbb{R}$. Let $\mathbb{G}_n$ and $\mathbb{P}_n^t$ denote the formal generator and the semigroup associated with $\omega^{(n)}$. If φ is a continuous and bounded cylinder function, then $\mathbb{P}_n^t \varphi \to \mathbb{P}^t \varphi$ as $n \to \infty$ follows from $\omega^{(n)} \to \omega$ for all $t > 0$ and $z \in \Omega$. Finally, if $\varphi \in \mathbb{C}_{0b}^2$ then the Ito lemma and (2.3) yield

$$\mathbb{P}_n^t \varphi = \varphi + \int_0^t \mathbb{P}_n^s \mathbb{G}_n \varphi \, ds$$

which completes the proof by letting n go to infinity. ▮

Our a priori bound for the stationary measures is based on

Lemma 3. Suppose that $\mu(\Omega) = 1$ and $\int \mathbb{G}\varphi \, d\mu = 0$ whenever $\mathbb{G}\varphi$ is well defined and bounded on Ω, then μ satisfies (1.14) ▮

Proof. Observe that if $\varepsilon > 0$ and $\delta > 0$ then

$$\mathbb{G}e^{-\varepsilon Q_m} = -\varepsilon e^{-\varepsilon Q_m} \mathbb{G} Q_m + \varepsilon^2 e^{-\varepsilon Q_m} \sum_{k \in S} r_k^2$$

is well defined and bounded on Ω. In view of (1.2) the quantity

$$M(\delta) = (T + K) \sup_{m \in S} \sum_{k \in S} e^{-\delta \rho(k,m)} < +\infty$$

for each $\delta > 0$, thus specifying δ as the unique solution to $4\delta K(1 + \lambda)\, e^{\delta} = \lambda$ we obtain from Lemma 1 that

$$\mathbb{G}e^{-\varepsilon Q_m} \geqslant \varepsilon \lambda [\tfrac{1}{4} W_m - M(\delta)]\, e^{-\varepsilon Q_m}$$

thus taking the expectation of both sides and dividing by $\varepsilon\lambda/4$ we have

$$\int W_m(\omega)\, e^{-\varepsilon Q_m(\omega)} \mu(d\omega) \leqslant 4M(\delta)$$

which completes the proof by letting ε go to zero ▮

Let us emphasize that the above bound does depend on λ.

Proposition 2. Every stationary measure of $\mathbb{P}^t$ satisfies (1.14). ▮

Proof. An easy limiting procedure yields

$$\mathbb{P}^t e^{-\varepsilon Q_m} = e^{-\varepsilon Q_m} + \int_0^t \mathbb{P}^s \mathbb{G} e^{-\varepsilon Q_m} \, ds$$

if $\varepsilon > 0$ and $\delta > 0$, and both sides are uniformly bounded on Ω, thus we can integrate with respect to μ; the result is just $\int \mathbb{G}e^{-\varepsilon Q_m} \, d\mu = 0$, which implies the statement by the argument of Lemma 3. ▮

Now we are in a position to manifest the ideas outlined at the end of Section 1. Many technical details follow Reference 4.

3. THE TEMPORAL DERIVATIVE OF FREE ENERGY

In addition to previous conditions, from now on we are assuming (1.6), too, while (1.4) and (1.5) are used mainly via (1.14) only. The notation introduced in the introduction and Section 1 will be used without any reference. Since $g_n(\omega, \cdot) \in \mathbb{C}^2_{0b}$, we have $\int \bar{\mathbb{G}} g_n(\omega, \bar{\omega})\, \mu(d\bar{\omega}) = 0$ for each $\omega \in \Omega$ if μ is a stationary measure of $\mathbb{P}^t$ (cf. Theorem 1 and Propositions 1 and 2). Consider now a twice continuously differentiable and bounded f: $[0, +\infty) \to [0, +\infty)$ such that $0 \leqslant f'(x) \leqslant 1/x$, then by means of (1.14) we obtain

$$\iint f\left(\frac{u_n(\omega)}{z_n(\omega)}\right) \bar{G} g_n(\omega, \bar{\omega})\, \mu(d\bar{\omega})\, d_n\omega = 0 \tag{3.1}$$

and the Fubini theorem allows us to change the order of integration. We shall let $f' \to 1/x$, then (3.1) will turn into (1.15). Our crucial trick is the following one. Since g_n is a function of $\omega - \bar{\omega}$, we have

$$\nabla_{\bar{p}k}\, g_n(\omega, \bar{\omega}) = -\nabla_{pk}\, g_n(\omega, \bar{\omega}) \quad \text{and} \quad \nabla_{\bar{q}k}\, g_n(\omega, \bar{\omega}) = -\nabla_{qk}\, g_n(\omega, \bar{\omega}) \tag{3.2}$$

thus we can replace the operators $\nabla_{\bar{p}k}$ and $\nabla_{\bar{q}k}$ appearing in $\bar{G}$ by ∇_{pk} and ∇_{qk}, respectively, therefore we can integrate (3.1) by parts. The case of p_k is really nice because $\mathbb{G}_p$ satisfies the equation of detailed balance, and unpleasant terms containing ∇_{qk} can be eliminated by means of the identity

$$(p_k - \bar{p}_k)\, \nabla_{qk}\, g_n(\omega, \bar{\omega}) = (q_k - \bar{q}_k)\, \nabla_{pk}\, g_n(\omega, \bar{\omega}) \tag{3.3}$$

More exactly, the contribution of one term of $\bar{\mathbb{G}}_p$ to the left hand side of (3.1) reads as

$$\begin{aligned} A_k(n, f) &= \iint f\left(\frac{u_n(\omega)}{z_n(\omega)}\right) [T\nabla^2_{\bar{p}k}\, g_n(\omega, \bar{\omega}) - \bar{p}_k \nabla_{\bar{p}k}\, g_n(\omega, \bar{\omega})]\, d_n\omega\, \mu(d\bar{\omega}) \\ &= -\iint f'\left(\frac{u_n}{z_n}\right)\left(\nabla_{pk} \frac{u_n}{z_n}\right) [T\nabla_{pk}\, g_n + \bar{p}_k\, g_n]\, d_n\omega\, \mu(d\bar{\omega}) \\ &= -TF_k(n, f) + R_{1k}(n, \sigma) \end{aligned} \tag{3.4}$$

where

$$F_k(n, f) = \int f'\left(\frac{u_n(\omega)}{z_n(\omega)}\right)\left(\nabla_{pk} \frac{u_n(\omega)}{z_n(\omega)}\right)^2 z_n(\omega)\, d_n\omega \tag{3.5}$$

$$R_{1k}(n, \sigma) = \iint f'\left(\frac{u_n(\omega)}{z_n(\omega)}\right)\left(\nabla_{pk} \frac{u_n(\omega)}{z_n(\omega)}\right)$$
$$\times (p_k - \bar{p}_k)\, g_n(\omega, \bar{\omega})\, d_n\omega\, \mu(d\bar{\omega}) \tag{3.6}$$

Indeed, putting $\bar{p}_k = p_k + (\bar{p}_k - p_k)$, the second expression, R_{1k} appears immediately. To recognize F_k, it is sufficient to notice that

$$\int \nabla_{pk}\, g_n(\omega, \bar{\omega})\, \mu(d\bar{\omega}) = \nabla_{pk} u_n(\omega)$$

and

$$p_k = -T\nabla_{pk} \log z_n(\omega) = -T\frac{\nabla_{pk} z_n(\omega)}{z_n(\omega)} \tag{3.7}$$

The step of integrating by parts will be justified a little bit later. The contribution of one term of $\mathbb{L}$ is

$$B_k(n, f) = \iint f\left(\frac{u_n(\omega)}{z_n(\omega)}\right) [\bar{p}_k \nabla_{\bar{q}k}\, g_n(\omega, \bar{\omega}) - c_k(\bar{\omega})\, \nabla_{\bar{p}k}\, g_n(\omega, \bar{\omega})]\, d_n\omega\, \mu(d\bar{\omega}) \tag{3.8}$$

where

$$c_k(\omega) = \nabla_{qk} H_k(\omega) = c_k(\omega, n) + b_k(\omega, n) \tag{3.9}$$

$$b_k(\omega, n) = \sum_{j \in S_k \setminus \Lambda_n} U'(q_k - q_j) \tag{3.10}$$

$$c_k(\omega, n) = -T\nabla_{qk} \log z_n(\omega) = -T\frac{\nabla_{qk} z_n(\omega)}{z_n(\omega)} \tag{3.11}$$

Using first (3.9), (3.2), (3.3) and the decompositions $\bar{p}_k = p_k + (\bar{p}_k - p_k)$, $c_k(\bar{\omega}, n) = c_k(\omega, n) + [c_k(\bar{\omega}, n) - c_k(\omega, n)]$, and then integrating by parts we obtain that

$$B_k(n, f) = TJ_k(n, f) + G_k(n, f) + R_{2k}(n, \sigma) + R_{3k}(n, \sigma) \tag{3.12}$$

where

$$J_k(n, f) = \int f\left(\frac{u_n}{z_n}\right)\left[\frac{\nabla_{qk} u_n}{z_n} \nabla_{pk} z_n - \frac{\nabla_{pk} u_n}{z_n} \nabla_{qk} z_n\right] d_n\omega$$
$$= \int \left[\left(\nabla_{qk} \bar{f}\left(\frac{u_n}{z_n}\right)\right) \nabla_{pk} z_n - \left(\nabla_{pk} \bar{f}\left(\frac{u_n}{z_n}\right)\right) \nabla_{qk} z_n\right] d_n\omega = 0$$

$$\bar{f}(x)=\int_0^x f(y)\,dy \tag{3.13}$$

$$G_k(n,f)=-\iint f'\left(\frac{u_n(\omega)}{z_n(\omega)}\right)\left(\nabla_{pk}\frac{u_n(\omega)}{z_n(\omega)}\right) \times b_k(\bar{\omega},n)\,g_n(\omega,\bar{\omega})\,d_n\omega\,\mu(d\bar{\omega}) \tag{3.14}$$

$$R_{2k}(n,\sigma)=-\iint f'\left(\frac{u_n(\omega)}{z_n(\omega)}\right)\left(\nabla_{pk}\frac{u_n(\omega)}{z_n(\omega)}\right) \times (q_k-\bar{q}_k)\,g_n(\omega,\bar{\omega})\,d_n\omega\,\mu(d\bar{\omega}) \tag{3.15}$$

$$R_{3k}(n,\sigma)=\iint f'\left(\frac{u_n}{z_n}\right)\left(\nabla_{pk}\frac{u_n}{z_n}\right)[c_k(\omega,n) - c_k(\bar{\omega},n)]\,g_n(\omega,\bar{\omega})\,d_n\omega\,\mu(d\bar{\omega}) \tag{3.16}$$

therefore (3.1) turns into

$$\lambda T\sum_{k\in\Lambda_n}F_k(n,f)=\sum_{k\in B_n}G_k(n,f)+\sum_{k\in\Lambda_n}\sum_{i=1}^{3}R_{ik}(n,\sigma) \tag{3.17}$$

The rigorous proof of (3.17) as well as its upper bound is given in the following lemma.

Lemma 4. We have a universal constant K such that

$$\lambda T\sum_{k\in\Lambda_n}F_k(n)\leqslant K\sum_{k\in B_n}[F_k(n)]^{1/2}+\frac{K}{\sigma}\sum_{k\in\Lambda_n}[F_k(n)]^{1/2}$$

for each n and $\sigma>0$. ∎

Proof. First we show that the quantities introduced above make sense. Equation (1.14) and the Cauchy inequality are the main tools. Observe that a logarithmic derivative is hidden in F_k, thus

$$F_k(n,f)=\int\frac{u_n}{z_n}f'\left(\frac{u_n}{v_n}\right)\left(\nabla_{pk}\log\frac{u_n}{z_n}\right)^2 u_n d_n\omega\leqslant F_k(n) \tag{3.18}$$

as $0\leqslant xf'(x)\leqslant 1$; the condition of equality is just $f(x)=\log x$. Moreover, the concrete form of g_n implies $u_n(\omega)>0$ for all $\omega\in\Omega$, and $|\nabla_{pk}u_n|\leqslant 9\sqrt{\sigma}\,u_n$, thus

$$F_k(n)=\int\left(\frac{\nabla_{pk}u_n}{u_n}+\frac{p_k}{T}\right)^2 u_n\,d_n\omega\leqslant 162\sigma+\frac{2}{T^2}\int p_k^2u_n\,d_n\omega$$

$$\leqslant 162\sigma+\frac{2}{T^2}\int\bar{p}_k^2\mu(d\bar{\omega})+\frac{4}{T^2}\iint(p_k-\bar{p}_k)^2\,g_n(\omega,\bar{\omega})\,d_n\omega\,\mu(d\bar{\omega})$$

$$\leqslant 162\sigma+\frac{4M}{T^2}+\frac{4c_2}{T^2\sigma^2} \tag{3.19}$$

where M is the universal constant coming from (1.14), while

$$c_2=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}c(p^2+q^2)(1+p^2+q^2)^{-9}\,dp\,dq$$

c is the normalizing factor of g_n.

Since $f'^2(u_n/z_n)\,u_n/z_n\leqslant f'(u_n/z_n)$, the Cauchy inequality yields

$$R_{1k}(n,\sigma)\leqslant F_k^{1/2}(n,f)\left[\iint(p_k-\bar{p}_k)^2\,g_n(\omega,\bar{\omega})\,d_n\omega\,\mu(d\bar{\omega})\right]^{1/2}$$

$$\leqslant\frac{2}{\sigma}\sqrt{c_2}\,F^{1/2}(n,f) \tag{3.20}$$

$$R_{2k}(n,\sigma)\leqslant\frac{2}{\sigma}\sqrt{c_2}\,F_k^{1/2}(n,f) \tag{3.21}$$

follows in the same way. The case of R_{3k} is a little bit more complicated, this is the point where (1.6) is needed. We have a family of random variables, $L_k=L_k(\omega)$, $k\in S$ such that

$$[c_k(w,n)-c_k(\bar{\omega},n)]^2\leqslant L_k(\bar{\omega})\sum_{j\in\Lambda_1(k)}[(q_j-\bar{q}_j)^2+(q_j-\bar{q}_j)^4] \tag{3.22}$$

and comparing (1.6) and (1.14) we see that $\int L_k(\bar{\omega})\,\mu(d\bar{\omega})\leqslant M$ for each $k\in S$, provided that M is large enough. Consequently

$$R_{3k}(n,\sigma)\leqslant[M\,\mathrm{card}\,\Lambda_1(k)(c_2\sigma^{-2}+c_4\sigma^{-3})]^{1/2}\,F_k^{1/2}(n,f) \tag{3.23}$$

where

$$c_4=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}cq^4(1+q^2+p^2)^{-9}\,dp\,dq$$

Finally, again by (1.4), (1.14) and by the Cauchy inequality

$$G_k(n,f)\leqslant F_k^{1/2}(n,f)\left[\int b_k^2(\bar{\omega},n)\,\mu(d\bar{\omega})\right]^{1/2}$$

$$\leqslant[M\,\mathrm{card}\,S_k\cap\Lambda_n]^{1/2}\,F_k^{1/2}(n,f) \tag{3.24}$$

In view of these estimates, to complete the proof it is sufficient to show that the boundary terms appearing at the steps of integrating by parts vanish at infinity. More exactly, collecting the analogous terms together, and exploiting $|\nabla_{pk} g_n| \leqslant 9\sqrt{\sigma}\, g_n$, $0 \leqslant f(x) \leqslant C$ and $0 \leqslant \bar{f}(x) \leqslant Cx$, these problems reduce to

$$\lim_{p_k \to \pm\infty} \iiint a_k(\omega, \bar{\omega})\, g_n(\omega, \bar{\omega}) \prod_{k \neq j \in \Lambda_n} dp_j \prod_{j \in \Lambda_n} dq_j\, \mu(d\bar{\omega}) = 0 \tag{3.25}$$

where $a_k(\omega, \bar{\omega}) = 1 + |\bar{p}_k| + |q_k - \bar{q}_k| + |b_k(\omega, n)| + |c_k(\omega, n) - c_k(\bar{\omega}, n)| + |c_k(\omega, n)|$, and to

$$\lim_{q_k \to \pm\infty} \iiint |p_k|\, g_n(\omega, \bar{\omega}) \prod_{j \in \Lambda_n} dp_j \prod_{k \neq j \in \Lambda_n} dq_j\, \mu(d\bar{\omega}) = 0 \tag{3.26}$$

It is plain that $g_n \to 0$ as $p_k \to \pm\infty$. Observe that g_n as a function of p_k attains its maximum at $p_k = \bar{p}_k$, while a_k does not depend on p_k, thus an easy calculation shows that the convergence of g_n mentioned above is actually a dominated one, which proves (3.25); the proof of (3.26) is the same.

Comparing the results of this section we obtain that

$$\lambda T \sum_{k \in \Lambda_n} F_k(n, f) \leqslant K \sum_{k \in B_n} F_k^{1/2}(n, f) + \frac{K}{\sigma} \sum_{k \in \Lambda_n} F_k^{1/2}(n, f) \tag{3.27}$$

Since $F_k(n, f) \leqslant F_k(n)$, (3.27) implies Lemma 4 by letting f go to $\log x$ in such a way that $f'(x) \to 1/x$, while $0 \leqslant xf'(x) \leqslant 1$ remains in force. ∎

Now we are in a position to conclude Theorems 2 and 3.

4. EVERY STATIONARY MEASURE IS A GIBBS STATE

We start with an easy consequence of Lemma 4.

Lemma 5. The conditions of Theorem 2 imply that $\liminf_{\sigma \to +\infty} F_k(n) = 0$ for each $k \in \Lambda_n$. ∎

Proof. Let $b_n = \text{card } B_n$, $c_n = \text{card } \Lambda_n$, and consider

$$X(n, \sigma) = \sum_{k \in B_n} F_k(n) \tag{4.1}$$

$$Q(n, \sigma) = \sum_{k \in \Lambda_n} F_k(n) \tag{4.2}$$

$$Z(n, \sigma) = F_\theta(0) + \sum_{r=1}^{n} X(n, \sigma) \tag{4.3}$$

Lemma 4 and the Cauchy inequality result in

$$\lambda T Q(n, \sigma) \leqslant K\sqrt{b_n}\, X^{1/2}(n, \sigma) + \frac{K}{\sigma}\sqrt{c_n}\, Q^{1/2}(n, \sigma) \tag{4.4}$$

whence, as $X(n, \sigma) \leqslant Q(n, \sigma)$, we obtain that

$$Q(n, \sigma) \leqslant \left[\frac{K}{\lambda \cdot T}\sqrt{b_n} + \frac{K}{\lambda\sigma T}\sqrt{c_n}\right]^2 \tag{4.5}$$

On the other hand, $u_n(\omega)$, $n = 0, 1, 2,...$ is a compatible family of probability densities, thus denoting by μ_σ the underlying probability measure, and taking into account that

$$F_k(n) = \int \left[\frac{\nabla_{pk} u_n(\omega)}{u_n(\omega)} + \frac{p_k}{T}\right]^2 \mu_\sigma(d\omega) \tag{4.6}$$

the Jensen inequality yields

$$F_k(r) \leqslant F_k(n) \qquad \text{if} \quad r \leqslant n \tag{4.7}$$

Therefore we have

$$\begin{aligned}(\lambda T)^2\, Z(n-1, \sigma)\, Z(n, \sigma) &\leqslant (\lambda T)^2\, Z^2(n, \sigma) \\ &\leqslant 2K^2 b_n[Z(n, \sigma) - Z(n-1, \sigma)] + R(n, \sigma)\end{aligned} \tag{4.8}$$

where $R(n, \sigma) \to 0$ for each n as $\sigma \to +\infty$ (see (4.5)). Suppose now that $\liminf_{\sigma \to +\infty} Z(n, \sigma) \geqslant \varepsilon > 0$ for $n \geqslant n_0$, then we have a $\sigma_0 > 1$ such that $Z(n, \sigma) > 0$ if $n \geqslant n_0$ and $\sigma > \sigma_0$, thus (4.8) turns into

$$\begin{aligned}(\lambda T)^2\, b_n^{-1} \leqslant\ & 2K^2[Z^{-1}(n-1, \sigma) - Z^{-1}(n, \sigma)] \\ &+ R(n, \sigma)\, b_n^{-1} Z^{-1}(n-1, \sigma)\, Z^{-1}(n, \sigma)\end{aligned} \tag{4.9}$$

for $n > n_0$, consequently

$$\begin{aligned}(\lambda T)^2 \sum_{n=n_0+1}^{N} b_n^{-1} \leqslant\ & 2K^2[Z^{-1}(n_0, \sigma) - Z^{-1}(N, \sigma)] \\ &+ \sum_{n=n_0+1}^{N} R(n, \sigma)\, b_n^{-1} Z^{-1}(n-1, \sigma)\, Z^{-1}(n, \sigma)\end{aligned} \tag{4.10}$$

Since $\sum b_n^{-1} = +\infty$, we can choose N to be so large that the left hand side of (4.10) exceeds $2K^2/\varepsilon$, but letting then σ go to infinity a contradiction is obtained, i.e., $\liminf_{\sigma \to +\infty} Z(n, \sigma) = 0$, whence the statement follows by (4.4) and (4.5). ▮

The translation invariant case is somewhat simpler, but the result is the same.

Lemma 6. Under the conditions of Theorem 3 we have $\liminf_{\sigma \to +\infty} F_k(n) = 0$ for each k and n. ▮

Proof. Condition (1.3) was used only at the end of the proof of Lemma 5, thus (4.4) and (4.5) can certainly be applied; we obtain

$$(\lambda T)^2 \sum_{k \in \Lambda_n} F_k(n) \leqslant 2K^2 b_n + 2K^2 \sigma^{-2} c_n \tag{4.11}$$

Since μ is translation invariant, we have $F_k(r) \leqslant F_j(n)$ whenever $\Lambda_r(jk^{-1}) \subset \Lambda_n$. Indeed, the finite-dimensional distribution of the variables (p_i, q_i), $i \in \Lambda_n(jk^{-1})$ is the same as that of (p_i, q_i), $i \in \Lambda_n$, thus the statement follows by the monotonicity property $F_i(r) \leqslant F_i(n)$, $r \leqslant n$, consequently

$$(\lambda T)^2 \, c_{n-2r-1} F_k(r) \leqslant 2K^2 b_n + 2K^2 \sigma^{-2} c_n \tag{4.12}$$

whence $(\lambda T)^2 F_k(r) \leqslant 2K^2 \sigma^{-2}$ follows by (1.2), which completes the proof of Lemma 6. ∎

The next step is to show that our μ has Maxwellian velocities of temperature T, i.e., the ratio $u_n(\omega)/z_n(\omega)$ becomes independent of p_k as $\sigma \to +\infty$.

Lemma 7. If $\liminf_{\sigma \to +\infty} F_k(n) = 0$ for each n, then we have

$$\int p_k \varphi(\omega) \, \mu(d\omega) = T \int \nabla_{pk} \varphi(\omega) \, \mu(d\omega)$$

for each bounded cylinder function φ such that $\nabla_{pk} \varphi$ is continuous and $\varphi(\omega) = 0$ if $|p_k|$ is larger than a certain constant. ∎

Proof. Since g_n is a singular integral

$$T \int \nabla_{pk} \varphi(\omega) \, \mu(d\omega) = \lim_{\sigma \to +\infty} T \int (\nabla_{pk} \varphi(\omega)) \, u_n(\omega) \, d_n\omega \tag{4.13}$$

if n is large enough, while

$$\begin{aligned} T \int (\nabla_{pk} \varphi(\omega)) \, u_n(\omega) \, \mu(d\omega) &= -T \int \varphi(\omega) \, \nabla_{pk} u_n(\omega) \, d_n\omega \\ &= \int p_k \varphi(\omega) \, u_n(\omega) \, d_n\omega \\ &\quad - T \int \varphi(\omega) \left(\frac{\nabla_{pk} u_n}{u_n} + \frac{p_k}{T} \right) u_n(\omega) \, d_n\omega \end{aligned} \tag{4.14}$$

therefore, as

$$\left|\int \varphi(\omega)\left(\frac{\nabla_{pk}u_n}{u_n}+\frac{p_k}{T}\right)u_n(\omega)\,d_n\omega\right| \leqslant \left[\int \varphi^2(\omega)\,\mu(d\omega)\right]^{1/2} F_k^{1/2}(n) \tag{4.15}$$

the statement follows by letting σ go to infinity along a sequence for which $F_k(n)\to 0$. ▮

Applying this lemma to the function $\varphi(\omega)\exp(p_k^2/2T)$ we get

$$\int (\nabla_{pk}\varphi(\omega))\exp(p_k^2/2T)\,\mu(d\omega)=0 \tag{4.16}$$

thus the following lemma implies that the conditional density of μ given q_k and (p_j, q_j) for $j\neq k$ is proportional to $\exp(-p_k^2/2T)$.

Lemma 8. If ρ is a σ-finite measure on $\mathbb{R}$ and $\int \varphi'(x)\,\rho(dx)=0$ for each continuously differentiable φ of compact support, then ρ is a multiple of the Lebesque measure. ▮

Proof. We have a nondecreasing function $\rho=\rho(x)$ such that $\int \varphi'(x)\,\rho(dx)=\int \varphi'(x)\,d\rho(x)=-\int \varphi''(x)\,\rho(x)\,dx=0$ whenever φ has two continuous derivatives, which is possible only if ρ is a linear function; see, e.g., the Weyl lemma in [11]. ▮

The conditional distribution of q_k will be found by means of

Lemma 9. If μ satisfies (1.14) and $\int \mathbb{G}\varphi(\omega)\,\mu(d\omega)$ for all $\varphi\in\mathbb{C}_{0b}^2$, then the conclusion of Lemma 7 implies that

$$\int (\nabla_{qk}H_k(\omega))\,\varphi(\omega)\,\mu(d\omega)=T\int \nabla_{qk}\varphi(\omega)\,\mu(d\omega)$$

whenever $\varphi\in\mathbb{C}_{0b}^2$ does not depend on p_k. ▮

Proof. In view of Lemma 7 we have $\int \mathbb{G}_p\varphi(\omega)\,\mu(d\omega)=0$ if $\varphi\in\mathbb{C}_{0b}^2$. Thus applying $\mathbb{G}$ to $\bar\varphi(p_k)\,\varphi(\omega)$, where $\varphi\in\mathbb{C}_{0b}^2$ does not depend on p_k, we obtain that

$$\int \bar\varphi'(p_k)(\nabla_{qk}H_k(\omega))\,\varphi(\omega)\,\mu(d\omega)=\sum_{j\in S}\int \bar\varphi(p_k)\,p_j\varphi(\omega)\,\mu(d\omega) \tag{4.17}$$

which implies the statement by letting $\bar\varphi(p_k)\to p_k$ in such a way that $\bar\varphi'\to 1$. Indeed, then $\int p_k\bar\varphi(p_k)\,\varphi(\omega)\,\mu(d\omega)\to T\int \varphi\mu(d\omega)$ and $\int p_j\bar\varphi(p_k)\,\varphi(\omega)\,\mu(d\omega)\to 0$ because (4.16) implies that p_k is independent of all of the other coordinates, and its probability density is just $(2\pi T)^{-1/2} \exp(-p_k^2/2T)$. ∎

Applying this result to $\varphi(\omega) \exp(H_k(\omega)/T)$, where $\varphi \in \mathbb{C}_{0b}^2$ does not depend on p_k, we obtain that

$$\int (\nabla_{qk} \varphi(\omega)) \exp\left(\frac{1}{T} H_k(\omega)\right) \mu(d\omega) = 0 \tag{4.18}$$

therefore Lemma 8 implies that the conditional density of q_k given (p_j, q_j) for $j \neq k$ is proportional to $\exp(-H_k(\omega)/T)$, which completes the proof of Theorems 2 and 3.

Remark 1. The crucial point of this version of the free energy method is the choice of the singular kernel g_n, in particular, (3.2), (3.3) and the boundedness of the logarithmic derivatives are the most relevant properties. There is another possibility, namely, the transition density of the partial dynamics $\omega^{(n)} = \omega^{(n)}(t)$ corresponding to $S = \Lambda_n$ can be used in place of g_n. In this case the correspondence between the forward and backward Kolmogorov equations plays the role of integration by parts. This method seems to be more general than the present one, but it is not easy to get the necessary bounds, partly because the Kolmogorov equations are degenerated parabolic equations in this case.

Remark 2. The second critical step is to find a suitable couple of Liapunov functions (Q, W) to derive the necessary moment conditions for arbitrary stationary measures. The present solution is based on the very particular structure of (1), and the presence of an at least quadratic external field is needed.

REFERENCES

1. C. Boldrighini, A. Pellegrinotti, and L. Triolo, *J. Stat. Phys.* **30**:123–155 (1983).
2. J. Fritz, *J. Stat. Phys.* **20**:351–369 (1979).
3. J. Fritz, *Colloquia Mathematica Societatis János Bolyai* 27, Random Fields, Esztergom (Hungary), 1979, (North Holland, Amsterdam, 1981).
4. J. Fritz, *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **59**:479–490 (1982).
5. B. M. Gurevich and Yu. M. Suhov, *Commun. Math. Phys.* **49**:63–96 (1976); **54**:81–96 (1977); **56**:225–236 (1977); **84**:333–376 (1982).
6. R. Holley, *Commun. Math. Phys.* **23**:87–99 (1971).
7. R. Holley and D. Stroock, *Commun. Math. Phys.* **55**:37–45 (1977).
8. R. Holley and D. Stroock, *J. Funct. Anal.* **42**:29–63 (1981).
9. A. N. Kolmogorov, *Math. Ann.* **113**:766–772 (1937).
10. O. E. Lanford, J. L. Lebowitz, and E. H. Lieb, *J. Stat. Phys.* **16**:453–461 (1977).
11. H. P. McKean, Jr., *Stochastic Integrals.* (Academic Press, New York/London, 1969).

12. A. Rényi, Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, I. (University of California Press, Berkeley/Los Angeles, 1960/61), p. 547–561.
13. H. Spohn, and J. L. Lebawitz, *Commun. Math. Phys.* **54**:97–121 (1977).
14. E. M. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, New Jersey, 1970).
15. M. M. Tropper, *J. Stat. Phys.* **17**:511–528 (1977).
16. A. M. Yaglom, *Mat. Sb.* **24**:457–492 (1949), (in Russian).